

# Renormalization Group on hierarchical lattices in finite dimensional disordered Ising and Blume-Emery-Griffiths Models

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Renormalization group analysis on hierarchical lattices is a valuable tool to understand the critical behavior of statistical mechanical models. In presence of quenched disorder, where position space renormalization group procedures are still on their way, it has the advantage to be analytically applicable. In some model cases, though, predictions obtained with the Migdal-Kadanoff bond removal approach fail to qualitatively reproduce critical properties obtained in mean-field approximation or by numerical simulations. We focus on the critical behavior of Ising and Blume-Emery-Griffiths models on hierarchical lattices microscopically more similar to finite dimensional Bravais lattices than those belonging to the Migdal-Kadanoff class in order to understand if and how different behaviors are associated with hierarchization.

## I. INTRODUCTION

Understanding the nature of the low temperature phase of spin-glasses, i.e., systems with quenched disordered magnetic exchange interaction, in finite dimensional systems has turned out to be an extremely difficult task. Since the resolution of its mean-field approximation, valid above the upper critical dimension ( $D = 6$ ), more than thirty years have passed without a final word about the possible generalization of mean-field properties of spin-glasses to finite dimensional cases. The mean-field, else called Replica Symmetry Breaking (RSB) theory<sup>1,2</sup> involves a very interesting solution for the spin-glass phase and its critical properties, rich of physical (and mathematical) implications, and has been fundamental in solving very diverse problems both in physics and in other disciplines.<sup>3-5</sup>

Because of its complicated structure, to overcome technical (maybe also conceptual) obstacles hindering the "portability" of RSB theory predictions to short-range systems on Bravais lattice in  $D < 6$  is a rather big challenge in theoretical physics. Indeed, the RSB solution is so complex that non-perturbative effects cannot be taken under control in any perturbative loop-expansion around the upper critical dimension and critical scaling behavior is yet to be understood.<sup>6-11</sup> The main hindrance is the lack of translational invariance in the position space for locally frustrated systems with quenched disordered interactions, making the techniques developed for quantum field theory and successfully exported to statistical mechanical problems<sup>12,13</sup> (e.g., for the Ising model critical exponents) inapplicable.

For what concerns Wilson's original approach in position space,<sup>14</sup> the extension of renormalization group techniques to disordered and locally frustrated systems is still on its way. The generalization of classic position space renormalization methods on Bravais lattices to disordered interaction, such as the ones proposed for Ising spin models in the seventies,<sup>15,16</sup> has led to controversial results. On the one hand, by means of a cumulant

expansion approach, evidence for a spin-glass phase is yielded in dimension two,<sup>17,18</sup> lower than the lower critical dimension  $D = 2.5$ .<sup>19-21</sup> On the other hand the renormalization through block transformation on spin clusters does not yield any spin-glass fixed point even in dimension three.<sup>17</sup>

The only consistent results have been achieved using "realisable" approximations,<sup>22</sup> namely those that are the exact solution of some alternative problem. So most of the studies have been concentrating on hierarchical lattices for which, in the ordered cases, the position space renormalization group flow is indeed exact (no truncation required). The study of these systems has brought to very important results, cf., e.g., Refs. [22-24] and Ref. [25] and references therein.

In this work we will investigate the renormalization group analysis on spin systems with quenched disorder on hierarchical lattices. We shall consider both Migdal-Kadanoff (MK) as well as more complex hierarchical lattices.

Our main aim is to investigate whether hierarchical cells locally more similar to short-range Bravais lattices can reproduce features of spin-glasses and ferromagnets so far unobserved in position space renormalization group studies, and then try to understand the limits and the advantages of this approach. As byproduct we will also provide new estimates for critical quantities on several hierarchical lattices.

As an instance, MK lattices fail to represent short-range spin-glasses on Bravais lattices also in the mean-field approximation and are, thus, strongly limited in probing the actual nature of the spin-glass phase.<sup>26,27</sup> For what concerns perturbative disorder in ferromagnetic systems, it has been found that Nishimori conjecture does not provide the exact value of the multicritical point coordinate,<sup>28,24</sup> although recently Ohzeki, Nishimori e Berker<sup>24,29</sup> have put forward an improved conjecture for models defined on hierarchical lattices and they found that for various families of these models a noteworthy recovery is obtained. Furthermore, MK lattices fail, even

qualitatively, in obtaining first order transitions both in ordered<sup>30</sup> and in disordered<sup>31</sup> systems. Eventually they do not show re-entrances observed in some spin-1 spin systems, e.g., the Blume-Emery-Griffiths models, both in mean-field theory<sup>32</sup> and by numerical simulations in finite dimension.<sup>31</sup> Our aim is to test whether and which of the above mentioned differences are specifically due to the bond moving procedure at the basis of the renormalization group analysis. We will, thus, implement and compare the analysis of the critical behavior of well known statistical mechanical models with quenched field and bond randomness on both MK hierarchical lattices and more complex hierarchical lattices, such as Wheatstone bridge (WB)<sup>33</sup> or folded-cell<sup>34</sup> lattices.

The paper is organized as follows: in Sec. II we expose the implementation of the position space renormalization group we used; in Sec. III we investigate Ising spin models, presenting and comparing the estimates of critical parameters and discussing how they comply to known statistical mechanics criteria in presence of disorder (Nishimori conjecture, Harris criterion, ferromagnetic line inversion, ...). We consider first, Sec. IIIB, the random field Ising model. Then, we study the Ising model with quenched bond disorder. In Sec. IIIC we analyze the behavior in low dimension, i.e.,  $d < 2.5$  where only paramagnetic and ferromagnetic phases are displayed, and, in Sec. IIID in  $d \geq 3$ , where the spin-glass phase arise. In Sec. IV we consider the Blume-Emery-Griffiths model on several hierarchical lattices in dimension  $d \geq 3$ . In the latter case our analysis shows a phase diagram displaying a reentrance for strong disorder, absent on MK lattices,<sup>35</sup> but present in the mean-field approximation<sup>36</sup> and numerical simulations on 3D cubic lattices.<sup>31,37,38</sup>

## II. HIERARCHICAL RENORMALIZATION

The Positional Space Renormalization Group (PSRG) approach, approximated on realistic Bravais lattices, become exact as it is iterated on Hierarchical Lattices (HL).<sup>22,34,39-41</sup> These lattices are constructed by carrying successive similar operations at each hierarchical level. E.g., at each level one replaces bonds by well-defined unit cells (see for example Fig. 1 for the diamond lattice or Fig. 2 for a MK lattice). The PSRG procedure works the inverse way of the lattice generation, i.e., through a decimation of the internal sites of a given cell, leading to renormalized quantities associated with the external sites.

In the pure case, the PSRG analysis proceeds as known by finding the interactions leaving the partition function invariant under decimation, and obtaining the critical exponents by the eigenvalues of the first derivative matrix computed on the relative fixed point.<sup>42</sup>

In disordered systems the PSRG transformation is described by the evolution of a probability distribution

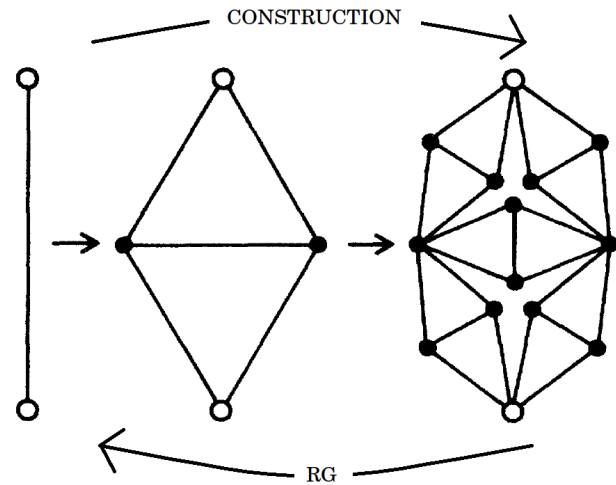


FIG. 1: Costruction of diamond, else called 2-dimensional Wheatstone-bridge, hierarchical lattice with fractal dimension  $d = \log 5 / \log 2 = 2.3219 \dots$

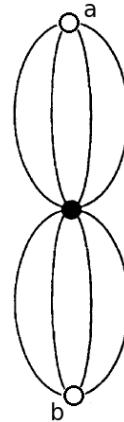


FIG. 2: *Necklace* MK lattice. It has  $b = 2$  and fractal dimension  $d = 3$ .

rather than single values of coupling constants:<sup>43,44</sup>

$$P'(\mathcal{J}') = \int \left[ \prod_{\alpha=1}^{b^d} d\mathcal{J}_\alpha P(\mathcal{J}_\alpha) \right] \delta[\mathcal{J}' - \mathcal{R}(\{\mathcal{J}\}_{b^d})] \quad (1)$$

where  $\mathcal{J}$  is the set of external parameters (couplings, fields, chemical potentials, ...),  $b$  is the the length of the cell in lattice spacings (i.e., the scaling factor in the decimation procedure),  $d$  the space dimension, so that  $b^d$  is the size of the cell in number of bonds to be decimated, and  $\mathcal{R}(\{\mathcal{J}\}_{b^d})$  is the local recursion relation for the interactions.

In MK lattices, because of their 1D-like topology, see, e.g., Fig. 2, the transformation can be divided into steps of so-called bond-moving and decimation, each of which

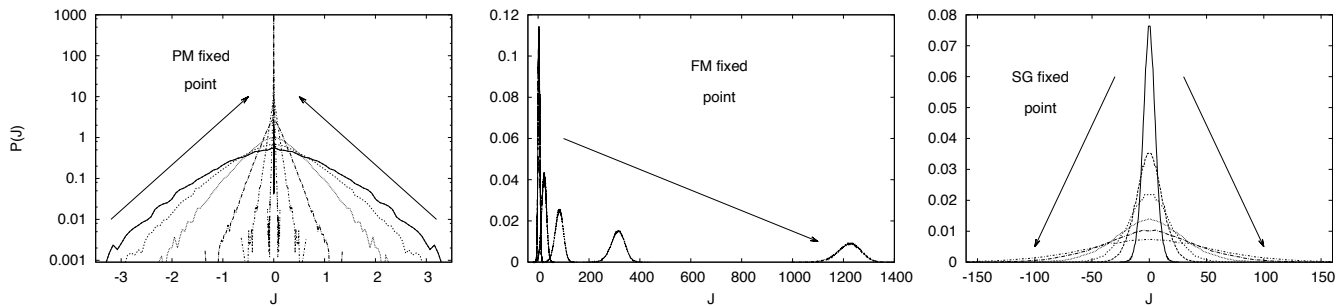


FIG. 3: RG evolution of the probability distribution of nearest-neighbor interaction for the spin-glass Ising model on the Wheatstone bridge lattice of Fig. 4 ( $D \simeq 3.58$ ) and pool size  $M = 10^6$ . Paramagnetic (left), Ferromagnetic (center) and Spin-Glass (right) phases.

involving only two bonds at a time. It is, thus, possible to exactly compute the probability distribution and represent it with histograms, each bin of which characterized by a value of the interactions and an associated probability.<sup>45</sup>

In HLs more complicated than MK, such as the Wheatstone-bridge-like in Fig. 1, this factorization is no longer possible and we must consider the convolution of more than two links at a time. We are, eventually, obliged to proceed in a statistical way. The PSRG scheme is, then, accomplished by representing the probability distribution of the couplings by a pool of  $M$  real numbers<sup>46</sup> from which one can compute its associated moments, at each renormalization step. In the limit  $M \rightarrow \infty$  these moments should approach those of the exact renormalized probability distribution. The process starts by creating a pool with  $M$  coupling constants generated according to the initial distribution. A PSRG iteration consists in  $M$  operations in which one randomly picks a set of  $b^d$  couplings from the pool in order to generate one renormalized coupling, which will populate the renormalized pool. Following this procedure, one creates a new pool of size  $M$  representing the renormalized probability distribution. During the PSRG procedure the moments of the coupling distribution are of particular interest for the identification of the phases.

For example, in Ising models with quenched disorder, by denoting with  $J$  the average of the couplings and with  $\sigma_J$  the mean square displacement, one obtains the Paramagnetic (PM), Ferromagnetic (FM), and Spin Glass (SG) phases, as dominated by the attractors

$$\begin{aligned} J &\rightarrow 0; \quad \sigma_J \rightarrow 0; & \text{PM;} \\ J &\rightarrow \infty; \quad \sigma_J \rightarrow \infty \quad (J/\sigma_J \rightarrow \infty); & \text{FM;} \\ J &\rightarrow 0; \quad \sigma_J \rightarrow \infty; & \text{SG;} \end{aligned}$$

In Fig. 3 the typical PSRG iterations of the couplings distribution in the three phases are shown for the random bond Ising model on the HL lattice of Fig. 4.

In order to reduce the dependence on a particular sequence of random numbers, the evolution of each distribution

is analyzed over  $N_S$  different samples. This is especially relevant when the starting pool is near a critical point, and random fluctuations can lead different samples of the same distribution into different attractors. In this case, we will adopt the convention that a phase, is considered as identified if at least 80% of the  $N_S$  samples flow into the same attractor. This defines the error for the location of critical points, which can be reduced by increasing the value of  $M$ .

### III. MODELS WITH ISING SPINS

The well known Ising model is defined by the Hamiltonian

$$-\beta\mathcal{H}(s) = J \sum_{\langle ij \rangle} s_i s_j + h \sum_i s_i, \quad (2)$$

where  $s_i = \pm 1$  and  $\langle ij \rangle$  indicates a sum over nearest-neighbor pairs. We stress that here and in the rest of the paper we include the temperature in the definition of the couplings (reduced parameters).

Since we, eventually, want to study the critical properties of disordered systems, and since through the renormalization group transformation original random bonds induce random fields (and vice-versa) already at the first step of renormalization, it becomes more convenient to start using the following Hamiltonian

$$-\beta\mathcal{H}(s) = \sum_{\langle ij \rangle} \left[ J_{ij} s_i s_j + h_{ij} \frac{s_i + s_j}{2} + h_{ij}^\dagger \frac{s_i - s_j}{2} \right] \quad (3)$$

In this way each link between two sites  $i$  and  $j$  is associated with three (possibly disordered) interactions  $J_{ij}$ ,  $h_{ij}$  and  $h_{ij}^\dagger$ .

Decimating the inner sites  $\{s\}$  of the basic cell  $\mathcal{C}_{ab}$  of the hierarchical lattice with external sites  $s_a$  and  $s_b$ , while imposing the conservation of the partition function of the

cell

$$Z_{C_{ab}} \equiv x_{s_a s_b} = \sum_{\{s\} \in C_{ab}} \exp \{ -\beta \mathcal{H}[s_a, s_b; \{s\}] \}, \quad (4)$$

yields the renormalization group equations:

$$\begin{aligned} J_R &= \frac{1}{4} \log \left( \frac{x_{++} x_{--}}{x_{+-} x_{-+}} \right), \\ h_R &= \frac{1}{2} \log \left( \frac{x_{++}}{x_{--}} \right), \\ h_R^\dagger &= \frac{1}{2} \log \left( \frac{x_{+-}}{x_{-+}} \right), \end{aligned} \quad (5)$$

The partition sums  $x_{s_a s_b}$ , also called principal Boltzmann factors of the cell, are the weights of the cell for fixed external spins  $s_a$  and  $s_b$ . The sum in Eq. (4) runs over all inner or free spins of the cell  $C_{ab}$ .

In the zero-temperature limit the relations become

$$\begin{aligned} 4J_R &= \max[-\beta \mathcal{H}(1, 1, s)] + \max[-\beta \mathcal{H}(-1, -1, s)] \\ &\quad - \max[-\beta \mathcal{H}(1, -1, s)] - \max[-\beta \mathcal{H}(-1, 1, s)] \\ 2h_R &= \max[-\beta \mathcal{H}(1, 1, s)] - \max[-\beta \mathcal{H}(-1, -1, s)] \\ 2h_R^\dagger &= \max[-\beta \mathcal{H}(1, -1, s)] - \max[-\beta \mathcal{H}(-1, 1, s)] \end{aligned} \quad (6)$$

When the external field is missing  $h = h^\dagger = 0$  and  $\mathcal{H}(s) = \mathcal{H}(-s)$ , which  $h_R = h_R^\dagger = 0$ .

### A. The ordered ferromagnetic Ising model

For an ordered ferromagnetic system ( $J_{ij} = J$ ,  $h_{ij} = h$  and  $h_{ij}^\dagger = h^\dagger$ ) the critical exponents can be obtained from the eigenvalues of the first derivatives matrix

$$\begin{pmatrix} \frac{\partial J_R}{\partial J} & \frac{\partial J_R}{\partial h} & \frac{\partial J_R}{\partial h^\dagger} \\ \frac{\partial h_R}{\partial J} & \frac{\partial h_R}{\partial h} & \frac{\partial h_R}{\partial h^\dagger} \\ \frac{\partial h_R^\dagger}{\partial J} & \frac{\partial h_R^\dagger}{\partial h} & \frac{\partial h_R^\dagger}{\partial h^\dagger} \end{pmatrix} \quad (7)$$

computed on the pure fixed point corresponding to the universality class of the ferromagnetic transition. The derivatives are easily obtained using

$$\begin{aligned} \frac{\partial x_{s_a s_b}}{\partial J} &= \sum_{\{s\} \in C_{ab}} \left( \sum_{\langle ij \rangle} s_i s_j \right) \exp[-\beta \mathcal{H}(s_a, s_b, \{s\})], \\ \frac{\partial x_{s_a s_b}}{\partial h} &= \sum_{\{s\} \in C_{ab}} \left( \sum_{\langle ij \rangle} \frac{s_i + s_j}{2} \right) \exp[-\beta \mathcal{H}(s_a, s_b, \{s\})], \\ \frac{\partial x_{s_a s_b}}{\partial h^\dagger} &= \sum_{\{s\} \in C_{ab}} \left( \sum_{\langle ij \rangle} \frac{s_i - s_j}{2} \right) \exp[-\beta \mathcal{H}(s_a, s_b, \{s\})]. \end{aligned} \quad (8)$$

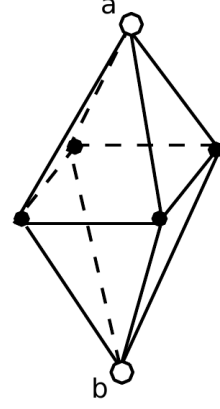


FIG. 4: Wheatstone bridge (WB) hierarchical lattice obtained by bonds moving from a cubic lattice. It has  $b = 2$  and fractal dimension  $d = \log 12 / \log 2 \approx 3.585$ .

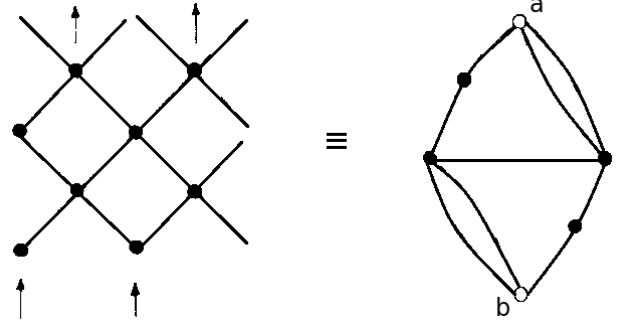


FIG. 5: Left hand side: square cells of cell spacing length  $b = 3$ . On the right side: corresponding *folded square* hierarchical lattices with two roots (open circles). All *outcoming* and *incoming* sites, pointed by the arrows, are put together and generate, respectively, root sites **a** and **b**. The inner sites of the square cell become the inner sites of the folded square hierarchical lattice. Note that with this construction we obtain self-dual lattices with fractal dimension two.

In particular, if the fixed point is for  $h = h^\dagger = 0$ , it is easy to see that the matrix in Eq. (7) is diagonal and  $\partial h_R^\dagger / \partial h^\dagger \equiv c$ , where  $c$  is the number of incoming (outgoing) link in the external outgoing (incoming) site. For example  $c = 4$  in Fig. 4,  $c = 3$  in Fig. 5 and  $c = 5$  in Fig. 6. In this case the only relevant eigenvalues are  $\lambda_T = \partial_T T' = \partial_J J'$  and  $\lambda_h = \partial_h h'$ , with the corresponding scaling exponents  $y_{T,h} = \log_b \lambda_{T,h}$ .

The above PSRG scheme holds when the coupling constants  $J_{ij}$  are all equal and the external field is ordered and homogeneous. To perform the equivalent analysis of the critical behavior in disordered systems one has to implement a PSRG method based on Eq. (1) and look for fixed point probability distributions. In this latter case, though, a general method for finding the critical exponents - like the one provided by Eqs. (7), (8) - is hard to be provided. The idea is, then, to estimate the criti-

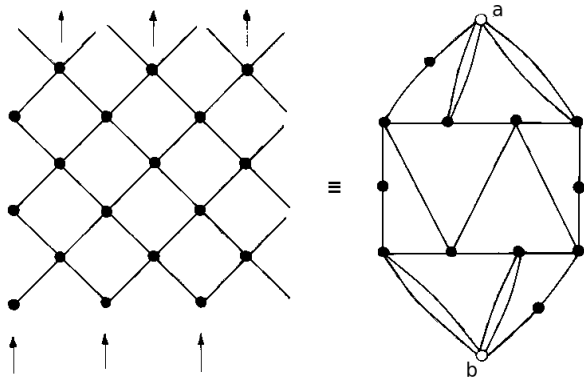


FIG. 6: Folded square lattice obtained as in Fig. 5, but for  $b = 5$ .

cal exponents by slightly perturbing the system from the unstable fixed point distribution and measure how *fast* it departs from it under successive PSRG iterations. We will discuss this procedure in detail in the following.

### B. Random Field Ising Model

In this section we discuss the PSRG study of the Random Field Ising Model (RFIM) with bimodal and Gaussian distributed quenched external field on the simple necklace MK lattice of Fig. 2, with fractal dimension  $d = 3$ , and on the Wheatstone-Bridge (WB) hierarchical lattice of Fig. 4, with fractal dimension  $d \approx 3.585$ . For the bimodal case on the MK lattice we recover known results.<sup>47,48</sup> The results on the WB lattice are, instead, obtained for the first time. We stress that in a previous paper of Nobre and Salmon<sup>33</sup> the exact PSRG transformation of the hierarchical lattice is not achieved, as pointed out by Berker.<sup>49</sup>

The initial distribution of couplings for the RFIM reads

$$P(J_{ij}, h_{ij}, h_{ij}^\dagger) = \delta(J_{ij} - 1) p(h_{ij}) \delta(h_{ij}^\dagger) \quad (9)$$

where  $p(h_{ij})$  is either a bimodal or a Gaussian distribution:

$$p(h_{ij}) = \begin{cases} \frac{1}{2} [\delta(h_{ij} - h_0) + \delta(h_{ij} + h_0)], \\ \frac{1}{\sqrt{2\pi h_0^2}} \exp\left\{-\frac{h_{ij}^2}{2h_0^2}\right\}. \end{cases} \quad (10)$$

The initial distribution is an even function of  $h$ , and  $h^\dagger$ . This symmetry is preserved under the PSRG transformation. To maintain this symmetry in our finite sample, we, actually, use a pool of  $2M$  interactions: we extract  $M$  values from the pool of renormalized interactions  $(J_{ij}, h_{ij}, h_{ij}^\dagger)$  and  $M$  from  $(J_{ij}, -h_{ij}, -h_{ij}^\dagger)$ . This trick becomes important for long PSRG iterations and as  $M$  increases the PSRG threshold step beyond which

it becomes necessary quickly increases. The forced symmetrization is, thus, not crucial in determining critical properties but helps decreasing finite size effects for small pools. For the present computation we take pools with up to  $M = 10^6$  interactions, enough to yield statistically stable results.

The RFIM shows two phases identified by the behavior of the ratio between the average coupling  $J = \langle J_{ij} \rangle$  and the standard deviation of the fields  $h_0^2 = \langle h_{ij}^2 \rangle$  along the PSRG flow:

$h_0/J \rightarrow \infty$  high temperature phase;

$h_0/J \rightarrow 0$  low temperature phase;

(and similarly for  $h^\dagger$  field). At the critical point both  $J$  and  $h_0$  (as well as  $\sigma_{h^\dagger}$ ) flow to infinity, confirming that the critical behavior of RFIM is controlled by a zero-temperature fix point couplings distribution.

For the MK lattice of Fig. 2 the critical point is obtained for the bimodal case at  $h_0 = 0.942(4)$ , with a fix point ratio  $h_0/J = 0.876(3)$ , whereas for the Gaussian case at  $h_0 = 0.934(5)$ , and fix point ratio  $h_0/J = 0.869(4)$ .

For the WB lattice shown in Fig. 4 the critical point is obtained in the bimodal case for  $h_0 = 0.460(3)$ , with a fix point ratio  $h_0/J = 0.443(5)$ , whereas for the Gaussian case at  $h_0 = 0.456(3)$ , and fix point ratio  $h_0/J = 0.448(4)$ .

In Fig. 7 we show the projections of the critical fix point probability distribution for the bimodal case on the WB hierarchical lattice. In the Gaussian case they are qualitatively the same.

At the zero-temperature fix point there are three independent exponents  $z$ ,  $y_h$  and  $y_T = 1/\nu$ ,<sup>50</sup> which can be obtained following the procedure of Ref. 47, for both the bimodal and Gaussian case.

*Distribution growth scaling.* The exponent  $z$  describes the deviation of the couplings probability distribution from the unstable fix point distribution under the PSRG transform. In order to estimate  $z$ , once the PSRG flux gets close to the fix point distribution, we fix the ratio  $h_0/J$  at the critical value in the subsequent PSRG iterations by shifting at each step all the couplings  $\{J_{ij}\}$  in the pool towards the ideal critical value. In particular, we choose to change them of the 80% of the difference with the critical value. In our computation each shift turns out to be very small, of the order of 0.1%, though essential because of the unstable nature of the fix point.

We, then, evaluate the rescaling factor  $\lambda \equiv h'_0/h_0 = J/J'$  at each PSRG step and then take its average. The exponent is then estimated as

$$z = \log_b \bar{\lambda}, \quad (11)$$

where in our case the overbar denotes the average over 10 PSRG steps, obtaining: on the MK cell, cf. Fig. 2,  $z = 1.491(3)$  for the bimodal and  $z = 1.486(3)$  for the Gaussian distribution; on the WB cell, cf. Fig. 4,  $z = 1.788(2)$  for the bimodal and  $z = 1.787(2)$  for the Gaussian distribution. We note that in all the cases it is

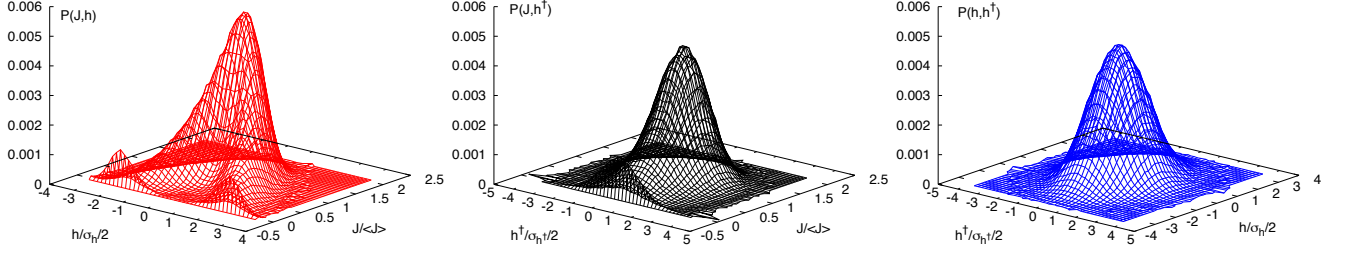


FIG. 7: Projections of the critical  $T = 0$  fixed probability distribution for RFIM on WB 2D lattice for bimodal initial distribution on  $J, h$  (left),  $J, h^\dagger$  (mid) and  $h, h^\dagger$  planes.

$z \leq d/2$ , that is the upper bound provided by Berker and McKay.<sup>51</sup>

*External field scaling.* The exponent  $y_h$  describes the rescaling of an infinitesimal homogeneous field and can be obtained by averaging the relations in Eq. (7), (8) over the fix point distribution

$$y_h = \log_b \left\langle \frac{\partial h_R}{\partial h} \right\rangle. \quad (12)$$

For MK we obtain  $y_h = 2.991(1)$  for the bimodal and  $y_h = 2.990(1)$  for the Gaussian distribution. For WB we obtain  $y_h = 3.575(1)$  for the bimodal and  $y_h = 3.576(1)$  for the Gaussian distribution. In all cases the value is smaller than the fractal dimension  $d$ , which is 3 for MK and  $\sim 3.585$  for WB, implying that the magnetization is continuous at the transition.

*Correlation length scaling.* In order to estimate the exponent  $\nu$ , we first reach a pool of renormalized couplings satisfactorily representing the fix point distribution. Next we take a copy of the pool and generate a slightly perturbed couplings probability distribution by shifting every coupling  $\{J_{ij}\}$  of the replicated pool by a small amount  $\delta = 10^{-4} \cdot J$ . The two pools, the original and the perturbed one, are then simultaneously renormalized. To reduce statistical fluctuations the couplings in the pool representing the fix point probability distribution are shifted, after each PSRG step, to keep the distribution close to the unstable fix point, in a manner similar to that used for the estimation of the exponent  $z$ . Note that in this way the only role of the shift  $\delta$  is to accelerate and make explicit the departure from the fixed point.

By defining  $t_n$  as the difference between the value of the ratio  $h_0/J$  in the two pools after  $n$  PSRG iterations, the correlation length exponent is estimated as

$$\frac{1}{\nu} = \log_b \left( \frac{\overline{t_{n+1}}}{t_n} \right), \quad (13)$$

where the overbar denotes the average over the PSRG iterations  $n$ . Note that the argument of the logarithm is always positive, because leaving the fix point the second

copy variance can either shrink or increase in its flux, but it does not oscillates between different PSRG steps.

Typically we have  $n \simeq 3 \div 9$ , for which the perturbed pool is not too far from the unstable fix point, and the ratio  $h_0/J$  is stable. The result is independent of  $M$  and it is quite stable over independent PSRG evolutions, at least for  $M \gtrsim 10^5$ .

We obtain  $1/\nu = 0.45 \pm 0.23$  for the bimodal distribution and  $1/\nu = 0.44 \pm 0.20$  for the Gaussian distribution on the MK lattice. Similarly, for the WB lattice we find  $1/\nu = 0.69 \pm 0.35$  for the bimodal distribution and  $1/\nu = 0.68 \pm 0.31$  for the gaussian distribution.

In Tab. III B we summarize the estimates of the critical exponents. Once the exponents  $z$ ,  $y_h$  and  $\nu$  are known, the exponents  $\alpha$  and  $\beta$  are obtained from the scaling relations<sup>50</sup>

$$\alpha = 2 - (d - z)\nu, \quad (14)$$

$$\beta = (d - y_h)\nu. \quad (15)$$

The critical exponents obtained on MK lattice for the bimodal case are compatible with those obtained by Cao and Machta.<sup>47</sup> The exponents  $y_h$  and  $z$  depend strongly on the fractal dimension of the lattice, and their value for the  $d = 3$  MK lattice are in good agreement with the results for the three dimensional Bravais lattice, whilst for the  $d \simeq 3.585$  WB lattice they are closer to that found for the four dimensional Bravais lattice, cf. Tab. III B.

The value of the exponent  $\nu$  is larger for the WB hierarchical lattice than for the MK lattice, as in agreement with the behavior in hyper-cubic lattices of increasing dimensions. Although the errors bars for this exponent are large, we stress that in this case only the WB lattice gives a numerical estimate compatible with the result for the cubic lattice.

We, eventually, notice that the Gaussian and bimodal exponents are always compatible with each other and belong to the same universality class.

In the RFIM the bond are at first all equal, and the disorder is on site. The major reason of moving from MK lattices to more complex HL is the need of a better treatment of the bond structure. So in the next sections we move to models in which the disorder is on the bond from the outset.

Lattice	$z$	$y_h$	$1/\nu$
bimod. MK	1.491(3)	2.991(1)	0.45(23)
Gauss MK	1.486(3)	2.990(1)	0.44(20)
bimod. WB	1.788(2)	3.575(1)	0.69(35)
Gauss WB	1.787(2)	3.576(1)	0.68(31)
3d MC <sup>52</sup>	1.49(3)	2.988(4)	0.73(5)
4d MC <sup>53</sup>	1.779(4)	3.827(1)	1.280(2)

TABLE I: Critical exponents at critical fixed distribution for RFIM on necklace MK lattice in Fig. 2 and WB lattice in Fig. 4. The last two rows relate to simulations on Bravais 3D and 4D hypercubic lattices with bimodal distributed interactions.

### C. Random Bond Ising model in $d \leq 2.5$

In this Section we consider the Ising model with bimodal  $\pm J$  bond distribution on hierarchical lattices mimicking the topology of the square lattice.

The initial probability distribution for interactions is

$$P(J_{ij}, h_{ij}, h_{ij}^\dagger) = [p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J)] \delta(h_{ij}) \delta(h_{ij}^\dagger), \quad (16)$$

where  $J > 0$  and  $p \in [0, 1]$  gives the probability of a ferromagnetic bond.

For low enough  $p$ , this model on regular lattice has an antiferromagnetic phase. On hierarchical lattices the antiferromagnetic order is preserved under PSRG only when the rescaling factor  $b$  is odd, so that a symmetric phase diagram in  $p \leftrightarrow (1-p)$  is obtained, where the ferromagnetic phase is replaced by the antiferromagnetic phase and viceversa.

To capture some features of Bravais lattices, we shall focus on hierarchical lattices with elementary cell more complex than the 1D-like MK cells. In particular we shall consider the cell proposed by Nobre,<sup>28</sup> Fig. 5, with rescaling factor  $b = 3$  and its extension to  $b = 5$ , Fig. 6.

*Nishimori conjecture.* Even though the Random Bond Ising model in 2D does not display any spin-glass phase, this model on hierarchical lattices is an excellent play ground to test Nishimori's conjecture.<sup>54,55</sup>

The idea behind the conjecture stems from noting that the partition function  $\mathcal{Z}$  of non-random Ising models on self-dual lattices is itself self-dual.<sup>56</sup> That is, if expressed in terms of the bond Boltzmann factors  $u_{\pm 1}(J) = e^{\pm J}$  (note the use of the reduced parameters),  $\mathcal{Z}$  is invariant under the exchange  $u_{\pm 1}(J) \leftrightarrow u_{\pm 1}^*(J)$ , where the  $u_{\pm 1}^*(J)$ , Fourier transforms of  $u_{\pm 1}(J)$ , are the dual Boltzmann factors.<sup>57</sup> As a consequence the critical point of a self-dual model is obtained, if unique, by the fix point condition  $u_{\pm 1}(J) = u_{\pm 1}^*(J)$ , which yields  $J_c = \frac{1}{2} \log(\sqrt{2} + 1)$ .

In the case of bimodal random bonds the relevant variables, within the replica method approach, are the averaged bond Boltzmann factors  $x_k(p, J)$ , which correspond

to the configuration with the spin connected by the bond equal to +1 in  $n - k$  replicas and -1 in remaining  $k$  replicas.

Self-duality is expressed by the invariance of  $\mathcal{Z}_n \equiv \overline{\mathcal{Z}}^n$  under the simultaneous exchange  $x_k(p, J) \leftrightarrow x_k^*(p, J)$  for all  $k$ . Unlike the non-random case, it is not possible to identify the critical point from the fix point condition of the duality relations, because the relations with  $k = 0, \dots, n$  are not satisfied simultaneously. The Nishimori's conjecture<sup>58,59</sup> identifies the multicritical point  $(p_N, J_N)$  with fix point condition for the leading  $k = 0$  Boltzmann factor

$$x_0(p_N, J_N) = x_0^*(p_N, J_N), \quad (17)$$

on the Nishimori line  $e^{-2J} = (1-p)/p$ ,<sup>54,60</sup> where enhanced symmetry simplifies the system properties significantly. This conjecture is proven exact for  $n = 1, 2$  and  $\infty$ . In the limit  $n \rightarrow 0$  the condition  $x_0 = x_0^*$  on the Nishimori line becomes

$$H(p_N) \equiv -p_N \log_2(p_N) - (1-p_N) \log_2(1-p_N) = \frac{1}{2}, \quad (18)$$

where the function  $H(p_N)$  is the binary entropy.

This conjecture has proved wrong for some systems on HL.<sup>24,61</sup> Ohzeki, Nishimori and Berker<sup>24,29</sup> noted that for HL a systematic approximation for the multicritical point can be obtained by imposing Eq. (17) at each PSRG transformations.

Note that in the two-dimensional case, even though the SG phase is absent, the Nishimori point is expected to coincide with a critical point, unstable along the phase boundary; so, as usual, in the following we identify the Nishimori point as the intersection between the Nishimori and the critical line.

Here we test of the original Nishimori conjecture on a Folded Square (FS) hierarchical lattice constructed through a bond-moving procedure that, at difference with the MK bond-moving prescription, retains the local correlation of the bonds, see Figs. 5 and 6.

#### 1. Numerical results

In Fig. 8 we show the  $p, T(= 1/J)$  phase diagram obtained from the PSRG analysis in two dimensions on the lattice shown in Fig. 6 with rescaling factor  $b = 5$ .

For  $p = 1$  we find the critical temperature is  $T_c = 2.269185(1)$ , in agreement with the Onsager solution  $T_c = 2/\log(1 + \sqrt{2}) \simeq 2.269185$ . This result, also found for the folded square lattice of Fig. 5 with  $b = 3$ ,<sup>28</sup> and WB with  $d = 2.32$ ,<sup>25</sup> follows from the duality properties of the unit cells.<sup>34</sup>

*Nishimori point.* The position of the multi-critical point expected from Nishimori's conjecture, Eq. (18), is  $p_N \simeq 0.889972$ . For the cell of Fig. 5 the estimated value is  $p_N = 0.8903(2)$  ( $2H(p_N) = 0.998(1)$ )<sup>24</sup> with  $T_N = 0.9557(18)$ .<sup>28</sup>



For the cell of Fig. 6, performing  $N_S = 20$  independent PSRG calculations with a pool of  $M = 10^5$  initial bond configurations, we estimate (cf. Tab. II)

$$p_N = 0.8902(1), \quad T_N = 0.9571(1) \\ 2H(p_N) = 0.9985(1);$$

thus we conclude that the conjecture fails also on this more complex hierarchical lattice.

The folded square cell estimates with  $b = 3$  or  $b = 5$  turn out to be in good agreement each other and with the estimate  $p_N = 0.8905(5)$  given by the transfer matrix approach,<sup>62</sup> but are slightly larger than those from high temperature series expansion,  $p_N = 0.886(3)$ ,<sup>63</sup> and Monte Carlo simulation,  $p_N = 0.8872(8)$ .<sup>64</sup>

In Tab. II we report the results on critical points for an easier comparison.

*Critical slope.* An interesting quantity to study is the slope of the critical line close to  $p = 1$ .<sup>65</sup>

$$s \equiv \frac{1}{T_c(1)} \left. \frac{dT_c(p)}{dp} \right|_{p=1}. \quad (19)$$

The Domany's perturbative approach<sup>66</sup> yields  $s = 2\sqrt{2}/[\ln(\sqrt{2}+1)] \simeq 3.209$  for the Ising with  $\pm J$  bond distribution on square lattice. The approach assumes *weak* disorder, i.e., a qualitatively irrelevant disorder that does not undermine the existence of the ferromagnetic phase at low  $T$  and does not change the universality class of the PM/FM transition. By computing  $s$  it is, thus, argued that one can discriminate whether quenched disorder is a relevant perturbation, causing a change in the universality class.<sup>67</sup> Ohzeki and collaborators<sup>67</sup> suggest that Domany's method can be applied to any self-dual lattice, and then one can probe the relevance of disorder from the slope  $s$  also for the HL of Figs. 5 and 6.

The duality approach gives  $s = 3.27866\dots$ <sup>67</sup> for the lattice in Fig. 5 with  $b = 3$ . In the  $b = 5$  case of Fig. 6, from a best fit of the points close to  $p = 1$  with a pool of size  $M = 5 \cdot 10^6$  with  $N_S = 20$  samples we find  $s = 3.316(11)$ . Both values are incompatible with the Domany's prediction  $s \simeq 3.209$ .<sup>66</sup>

According to the argument of Ref. [67] this would imply that the disorder should be relevant in these cases. However, if it is true that a slope equal to Domany's slope would be consistent with the hypothesis of irrelevant disorder, the fact that the slope is different from Domany's value does not, actually, imply the opposite (i.e., the existence of a new "strong disorder" fixed point). Indeed, from a direct inspection, no other fix points are found because of the introduction of disorder and consequently the universality class does not change. Any coupling distribution on the transition line tends to become more and more peaked under PSRG transformations, tending to the pure FM-PM critical fixed point, as shown in Fig. 9. Disorder in the two dimensional Ising model does not appear to play any role on any lattice cell analyzed here (see also the Harris criterion below).

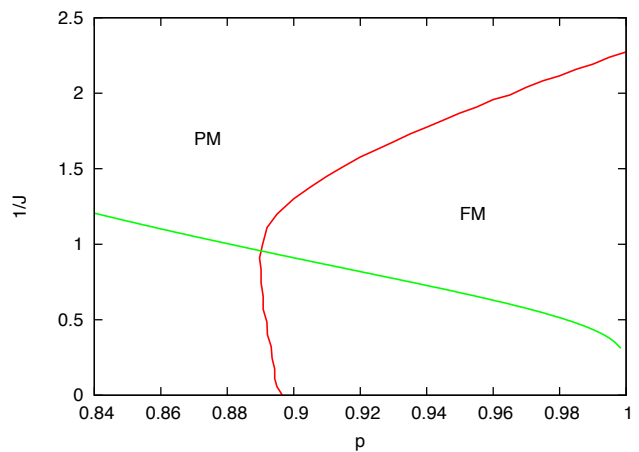


FIG. 8: Phase diagram of Ising 2D with "folded square" HL with  $b = 5$ . The diagram is symmetric in  $p \rightarrow 1 - p$  and the anti-ferromagnetic part is not shown. The dashed line represents the Nishimori line.

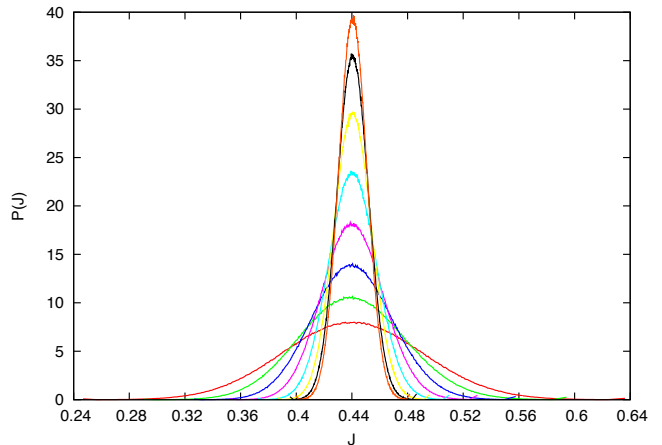


FIG. 9: PSRG evolution of the coupling distribution on the critical line near the pure fixed point (steps from 4 to 11 are shown for  $p = 0.01$ ): the width quickly decreases, so that the distribution tends to a Dirac delta and the disorder disappears. No other fixed ditribution, besides the pure criticality, are obtained on the PM-FM transition line.

HL / Method	Fig./Ref.	$p_N$
FS $b = 3$	5/[28]	0.8903(2)
FS $b = 5$	6	0.8902(1)
Square	[62]	0.8905(5)
	[63]	0.886(3)
	[64]	0.8872(8)
Nishimori conjecture	[54]	0.889972...

TABLE II: Multicritical point as computed on different HLs and different estimates on the square lattice.



*Harris criterion.* The widely accepted form of the Harris criterion<sup>65</sup> is that in ferromagnetic systems with random interactions the randomness is irrelevant if  $\alpha$ , the specific heat exponent of the corresponding pure system, is negative, while for systems with positive  $\alpha$  the random system exhibits different critical behavior in presence of disorder. For the folded square lattices for both  $b = 3$  and  $b = 5$  we find  $\alpha < 0$ .

Though it is known that this criterion can fail on HL if the bonds in the rescaling volume are not all equivalent,<sup>44,68–72</sup> in the present case the Harris criterion confirms that the disorder is, actually, irrelevant, as discussed above with respect to the critical slope value.

*FM line reentrance.* An important feature of the  $p, T$  diagram, according to the duality requirements, is the reentrance of the transition line below the multicritical point:  $p_N > p_{T=0}$ . The zero temperature transition point  $p_{T=0}$  can be estimated by finite size scaling analyses of the ground state. Calling  $E_p$  and  $E_a$  the ground-state energies with, respectively, periodic and anti-periodic boundary conditions in one direction, and  $\Delta = E_p - E_a$  the domain wall energy, we can determine two estimates of critical concentrations of antiferromagnetic bonds by looking at the point where the asymptotic dependences  $[\Delta]$  and  $[\Delta^2]^{1/2}$  change from increasing to decreasing, where the average [...] is taken on different bond samples. More explicitly, by defining

$$[\Delta] \sim L^\rho \quad \text{and} \quad [\Delta^2]^{1/2} \sim L^\theta,$$

via the determination of exact ground states for large system sizes and huge sample numbers, Kawashima and Rieger give the estimates  $p_{T=0}^{(1)} = 0.896(1)$  and  $p_{T=0}^{(2)} = 0.894(2)$  looking at the point where  $\rho$  and  $\theta$  change sign.<sup>73</sup> The PSRG approach with the folded square of Fig. 5 ( $b = 3$ ) leads to  $p_{T=0} = 0.8951(3)$ ,<sup>28</sup> while with the cell of Fig. 6 ( $b = 5$ ) we find  $p_{T=0} = 0.8966(2)$ .

The critical indices for the pure ferromagnetic point can be evaluated as discussed in Sec. III A: we obtain  $y_T = 0.7303(1)$  and  $y_h = 0.8518(1)$  for the folded square lattice with  $b = 3$ , while for the folded square lattice with  $b = 5$  we find  $y_T = 0.7589(1)$  and  $y_h = 1.059(1)$ . In both cases they are not consistent with the values of Onsager solution, which read  $y_T = 1$  and  $y_h = 1.875$ , respectively. From the knowledge of  $y_T$  and  $y_h$  the critical indexes are obtained from the usual scaling relations:

$$\nu = \frac{1}{y_T}, \quad (20)$$

$$\eta = d + 2 - 2y_h \quad (21)$$

$$\alpha = 2 - \frac{d}{y_T}, \quad (22)$$

$$\beta = \frac{d - y_h}{y_T}, \quad (23)$$

$$\gamma = \frac{2y_h - d}{y_T}, \quad (24)$$

$$\delta = \frac{y_h}{d - y_h}. \quad (25)$$

	FS b=3	FS b=5	Exact
$\alpha$	-0.7385(1)	-0.6353(1)	0
$\beta$	1.572(1)	1.240(1)	0.125
$\gamma$	-0.4057(1)	0.1558(1)	1.75
$\delta$	0.7419(1)	1.126(1)	15
$\nu$	1.369(1)	1.318(1)	1
$\eta$	2.296(1)	1.882(1)	0.25

TABLE III: Critical indices of pure ferromagnetic critical point for Ising model on the folded square lattice for  $b = 3$ , Fig. 5, and for  $b = 5$ , Fig. 6.

Lattice type	Fig. Ref.	$\theta$
MK $b = 2$	[46]	-0.270(2)
MK $b = 3$	[76]	-0.278(2)
WB $b = 2$	1 [34]	-0.290(3)
WB $b = 3$	[76]	-0.298(2)
FS $b = 3$	5 [34]	-0.275(1)
FS $b = 5$	6	-0.2714(2)
Square	[77]	-0.291(2)
	[78]	-0.287(4)
	[79]	-0.284(4)

TABLE IV: Stiffness exponent  $\theta$  on different HLs and different estimates on the square lattice.

and their numerical values are reported and compared on Tab. III. Notice that, at difference with Eq. (14), here the fixed point is at finite  $T$  (as  $1/J$ ) and the index  $z = 0$ .<sup>74</sup>

*Zero temperature stiffness.* We conclude this section by discussing the exponent  $\nu$  of the zero-temperature spin-glass transition for the case of a Gaussian distribution of bonds with zero mean and initial width  $\sigma_J$ . It can be obtained directly from scaling of  $\sigma_J$  under PSRG :

$$\sigma'_J(b) \sim \sigma_J b^\theta. \quad (26)$$

The sign of the stiffness exponent  $\theta$  is directly related to the low temperature phase: for positive (negative)  $\theta$  the system scales under PSRG flow towards strong (weak) couplings, distinctive of a low temperature spin-glass (paramagnetic) phase. For continuous and symmetric probability distributions  $P(J)$ , the temperature  $T$  appears in the PSRG equations as a dimensionless ratio between couplings, so that the scaling (26) is equivalent to  $T \sim L^\theta$ , or  $L \sim T^{1/\theta}$ . In a phase transition at  $T \rightarrow 0$  the latter scaling can be identified with the scaling of the correlation length  $\xi \sim T^{-\nu}$  implying:<sup>75</sup>

$$\nu = -1/\theta. \quad (27)$$

In Fig. 10 it is shown the behavior of  $\sigma_J$  as function of PSRG step for the case of zero-average Gaussian initial bond distribution on the folded square cell of Fig.

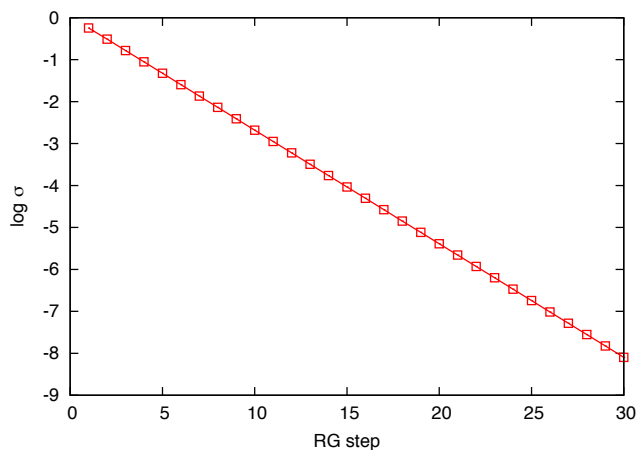


FIG. 10: Standard deviation of the Gaussian probability distribution of quenched disorder at  $T = 0$  vs. PSRG iteration steps.

6. From this we get the  $\theta = -0.2714(2)$ , leading to  $\nu = 3.685(3)$ . For the cell of Fig. 5, with  $b = 3$  it was found  $\theta = -0.275(1)$ .<sup>76</sup> We report in Tab. IV the values obtained on different HLLs<sup>25,34,46,76</sup> and on the square Bravais lattice.<sup>77–79</sup> We observe that the value for the folded square with both  $b = 3$  or  $b = 5$  is similar to that found for the MK lattice, whilst for the WB lattices the values are closer to that of the regular lattice, especially the case with  $b = 2$ .

Passing from  $b = 3$  to  $b = 5$ , and thus increasing the connectivity of the lattice, we obtain an improvement, as compared to square lattice, only for the pure model (although the estimates for the critical exponents are still not compatible with the exact values of the square lattice), whilst the values for the disordered quantities are generally compatible with each other.

To obtain tests for quantities with relevant disorder we have to move to models in which there is a spin-glass phase, which we will analyze in the following sections.

#### D. Random bond Ising model in $d \geq 3$

In this section we consider the Ising model with the  $\pm J$  coupling distribution, Eq. (16) on hierarchical lattices mimicking the topology of the cubic lattice. Recently, Salmon, Agostini and Nobre have studied the Ising spin glass on the hierarchical Wheatstone bridge (WB) lattices drawn in Figs. 1, 4,<sup>25</sup> obtaining accurate phase diagrams and showing that on these lattices the lower critical dimension for the spin glass phase is greater than  $d = \ln 5 / \ln 2 \simeq 2.32$ . The next pattern of the WB family, cf. Fig. 4, corresponding to a lattice in dimension higher than 2.5 has a fractal dimension  $d \approx 3.58$ . Large deviations from the cubic lattice behavior might, then, occur and results somewhat in the middle between 3D and 4D Bravais lattice models.

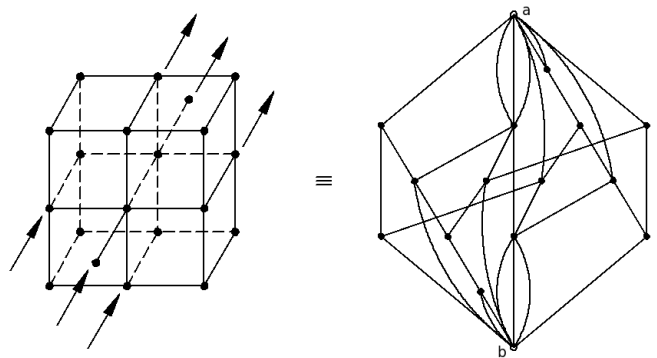


FIG. 11: Cubic cells of length  $b = 3$  on the left hand side, and relative folded cube hierarchical lattice on the right hand side. The two roots are open circles. All *incoming* sites and all *outcoming* ones, pointed by the arrows in the left figure, are put together and generate root sites 1 and 2 on the right hand side figure. The inner sites of the cubic cell become the inner sites of the hierarchical cell.

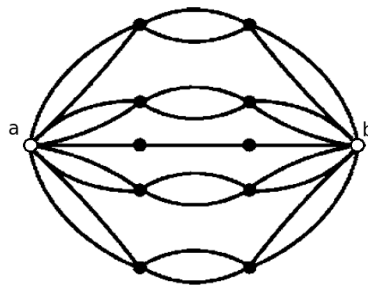


FIG. 12: MK lattice with  $b = d = 3$ .<sup>61,81</sup>

We, here, investigate the model on the Folded Cube (FC) hierarchical lattice shown in Fig. 11. This is the “three-dimensional” extension of the folded square lattice of Fig. 5, and it was introduced to study the anisotropic ferromagnetic Potts model in three dimensions.<sup>80</sup> This lattice has a fractal dimension  $d = \ln 35 / \ln 3 \simeq 3.2362$ , always larger than 3, though nearer to it as compared to the WB. It has  $b = 3$  and, unlike the latter, it is able to retain a possible antiferromagnetic order, as well. As a further comparison we will report on the critical properties of the Ising spin-glass model on a MK lattice with  $b = 3$  and fractal dimension  $d = 3$ , cf. Fig. 12, introduced in Refs. [35,81].

The resulting phase diagram is shown in Fig. 13. An important feature of the phase diagram is the small reentrance in the region below the multicritical point. As a consequence, by lowering the temperature one goes from the high temperature (disordered) paramagnetic (PM) phase to an ordered ferromagnetic (FM) phase and, eventually to a low temperature disordered spin-glass (SG) phase. The values of the critical points are reported in Tab. V.

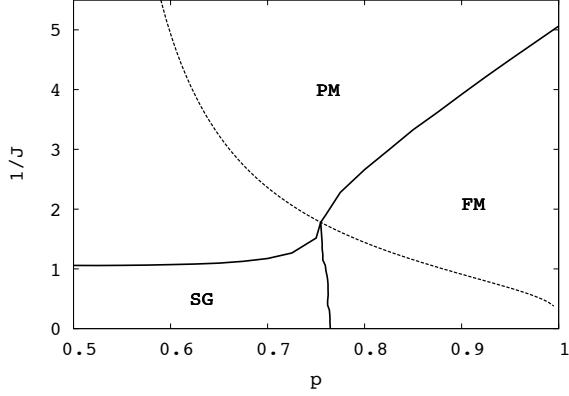


FIG. 13: Phase diagram of Ising spin glass model on the folded cube HL, cf. Fig. 11, obtained for  $M = 5 \cdot 10^4$  e  $N_S = 10$ . The dashed line represents the Nishimori line. The plot is symmetric in  $p \rightarrow 1 - p$  and an antiferromagnetic phase is present a small  $p$ , in place of the FM one.

Fixed points  $(p, T)$  coordinates

HL	FM	SG	$T = 0$	MC
Fold cube Fig. 11	1 5.066(1)	0.5 1.072(1)	0.764(2) 0	0.7547(3) 1.779(1)
WB "3D" Fig. 4	1 5.457(1)	0.5 1.112(2)	0.760(1) 0	0.745(2) 1.620(2)
WB "4D" Ref. [25]		0.5 2.515(2)	0.667(2) 0	0.664(2) 2.836(2)
MK Fig. 12	1 5.383(1)	0.5 1.136(4)	0.761(1) 0	0.752(7) 1.797(3)

TABLE V: Position of typical critical points of phase diagram in Fig. 13 for the hierarchical lattice 11: ferromagnetic (FM), spin-glass (SG), zero temperature ( $T = 0$ ) and multi critical point (MC). For comparison also values of the same fixed points are reported for other hierarchical lattices discussed in the text.

### 1. FM fixed point

The transition for  $p = 1$  on the folded cube cell is obtained at  $T_c = 5.066(1)$ , 12% larger as compared to the same model on cubic lattice (where  $T_c = 4.5115...$ <sup>82</sup>). On the  $d \simeq 3.58$  WB lattice the difference was about 21%.<sup>25</sup> On the MK lattices the best known result is obtained for the  $d = 3$  lattice in Fig. 12, where the critical temperature of the pure transition is  $T_c = 5.38(3)$ , 19% larger than the one in the cubic lattice.<sup>81</sup>

In the folded cube the points on the critical line between the FM and the PM phases are attracted to a single fix point located at  $p = 1$ . The critical exponents at

critical index	MK Fig. 12	WB Fig. 4	Folded cube Fig. 11	Cubic Ref. 84
$\alpha$	-0.2169(1)	-1.121(1)	-0.1253(1)	0.110(1)
$\beta$	1.112(1)	2.282(1)	0.9012(1)	0.3265(3)
$\gamma$	-0.006490(1)	-1.443(1)	0.3229(1)	1.2372(5)
$\delta$	0.9942(1)	0.3676(1)	1.358(1)	4.789(2)
$\nu$	0.6850(1)	0.8705(1)	0.6567(1)	0.6301(4)
$\eta$	2.009(1)	3.658(1)	1.508(1)	0.0364(5)

TABLE VI: Critical exponents in pure ferromagnetic critical point for Ising model on the folded cubic cell and comparison.

this point are  $y_T = 1.523(1)$  and  $y_h = 1.864(1)$ , identifying a second order transition.<sup>83</sup> On other HLs we obtain:  $y_T = 1.149(1)$  and  $y_h = 0.9636(1)$  for the WB lattice in Fig. 4;  $y_T = 1.460(1)$  and  $y_h = 1.613(1)$  for the MK lattice in Fig. 12. These are to be compared with numerical estimates by means of simulations on the cubic lattice:  $y_T = 1.587(1)$  and  $y_h = 2.482(1)$ .<sup>84</sup> Though none of them agrees with the cubic lattice results the Folded cube lattice in Fig. 11 yields the nearest estimate for both exponents. The corresponding physical exponents are reported in Table VI. Note, in particular, that for all the HLs considered here we have  $\alpha < 0$  and the disorder is irrelevant, thus satisfying the Harris criterion<sup>65</sup>: the distributions on the critical PM-FM line flow to the pure model critical point.

### 2. Spin-glass fixed point

For the DC HL in the totally disordered case  $p = 0.5$  the PM-SG transition is located at  $T_c = 1.072(1)$ , with a difference of 4.5% relatively to the Bravais cubic lattice critical point, for which the Monte Carlo simulations provide  $T_c = 1.120(4)$ .<sup>85</sup> As reported in Tab. V the WB lattice in Fig. 4 has  $T_c = 1.112(2)$  and the MK lattice in Fig. 12 has  $T_c = 1.136(4)$ .

The behavior of  $\ln(\sigma_J)$  under successive PSRG iterations for  $p = 0.5$  is shown in Fig. 14. From its linear behavior we estimate for the stiffness exponent of the spin-glass phase  $\theta = 0.2052(1)$  (cf. Sec. III C 1). For the cubic lattice one finds in the literature  $\theta = 0.19(1)$ <sup>77</sup> or  $\theta = 0.20(5)$ .<sup>86</sup> We, thus, obtain a result very close to that expected for the cubic lattice. On the WB lattice one has  $\theta = 0.297(3)$ ,<sup>25</sup> while for the MK lattice it is found  $\theta \approx 0.27$ .<sup>86</sup> The result are summarized in Table VII.

All critical points along the PM-SG transition are attracted by an unstable fixed point at  $p = 0.5$ . In Fig. 15 we show the unstable fix point coupling distributions for the three lattices considered so far. Critical exponents for this (unstable) fix distribution are computed by adapting the methods used for the RFIM in Sec. III B. A first exponent  $y_h$  can be obtained generalizing the pure case definition:  $\lambda_h = \langle \partial_h h_R \rangle$ , where the average is carried

Lattice type	Fig. Ref.	$\theta$
MK $d = 3$	12 [86]	0.27(1)
WB $d \sim 3.58$	4 [25]	0.297(3)
FS $d \sim 3.24$	11	0.2052(1)
Cubic	[77]	0.19(1)
	[86]	0.20(5)

TABLE VII: Stiffness exponent as computed on different HL and different estimates on the cubic lattice.

out by extracting the interactions from the fix point distribution. Indeed, in our case this is easy and can be exactly calculated. In fact for the fix point distribution we have  $h = h^\dagger = 0$ , and so  $\mathcal{H}(s) = \mathcal{H}(-s)$ . It is, then, straightforward to obtain, using Eqs. (7) and (8) that  $\partial_h h_R = c$  for every choice of the  $J$  interactions, where  $c$  is the number of internal sites connected to each external site:  $c = 9$  for the folded cube cell in Fig. 11 and the MK in Fig. 12,  $c = 4$  for the WB in Fig. 4. For all these cells the exponent  $y_h$  is then  $y_h = \log_b \langle \partial_h h_R \rangle = 2$  (see Sec. III A). We note that  $y_h < d$ , ensuring that the magnetization is continuous at the transition (no first order transition).

To get the correlation length exponent  $\nu$  we note that defining  $t \equiv \sigma_J - \sigma_J^*$ , where  $\sigma_J^*$  is the value at the critical point, we obtain the scaling law (note that, as always in this paper, we are using the reduced variables  $\beta J \rightarrow J$ )

$$\xi(t) = b \xi(t') = b \xi(b^x t) = b^n \xi(b^{nx} t). \quad (28)$$

By taking  $n$  such that  $b^{nx} t = t_0$ , where  $t_0$  is arbitrary but fixed, we end up with

$$\xi(t) = \left( \frac{t_0}{t} \right)^{\frac{1}{x}} \xi(t_0) \sim t^{-\frac{1}{x}}, \quad (29)$$

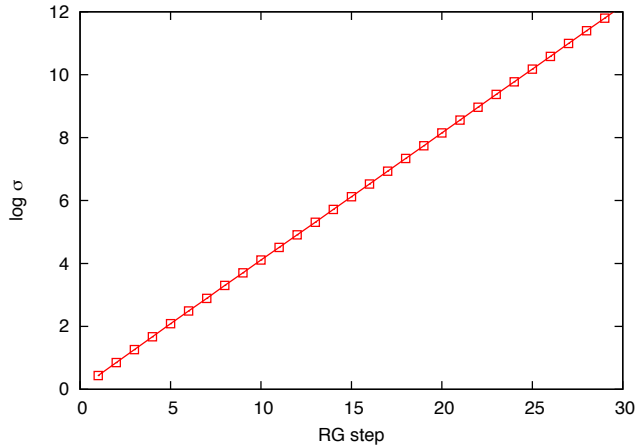


FIG. 14: Graph of  $\ln(\sigma_J)$  under repeated applications of the PSRG transformation at zero temperature and  $p = 0.5$  on lattice in Fig. 11.

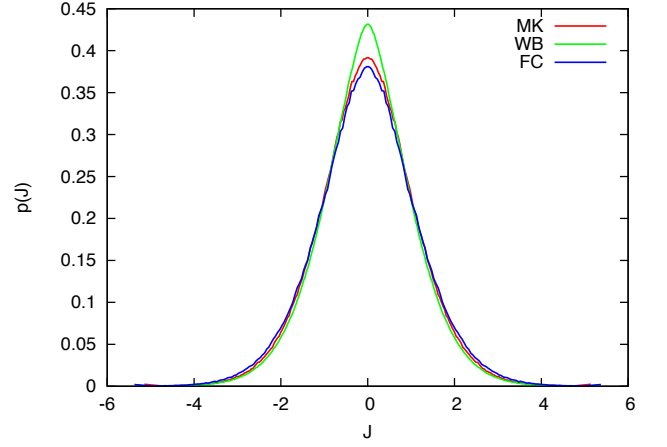


FIG. 15: Fixed probability distribution for the spin glass - paramagnet transition for Ising spin glass on lattice in Fig. 12 (MK), Fig. 4 (WB) and Fig. 11 (FC). These distributions are reached after about 2–3 steps of renormalization, starting from the binomial distribution, and maintain this shape for about 10 – 15 steps, after which they starting moving away because of statistical fluctuations.

Lattice	Fig.	Resc.	Dim.	$1/\nu$
MK	12	$b = 3$	$d = 3$	$0.297 \pm 0.026$
WB	4	$b = 2$	$d \sim 3.58$	$0.308 \pm 0.063$
Folded cube	11	$b = 3$	$d \sim 3.24$	$0.262 \pm 0.037$
Bravais cubic <sup>87</sup>			$d = 3$	$0.408 \pm 0.025$

TABLE VIII: Estimates for the exponent  $\nu$  at transition SG-para for the Ising spin glass model.

from which we get  $\nu = 1/x$ : the value of  $\nu$  can be estimated by the trend of  $\sigma_J$  near the critical point distribution.

We use the method already exposed in Sec. III B: after obtaining an instance of the fixed distribution, we make a copy of it and multiply each coupling of the copy by a quantity  $(1 + \delta) \approx 1.05$ . The two copies of the distribution are, then, simultaneously iterated in the PSRG transformation, with the first copy forced near the fixed point and the second one free to flow. We eventually estimate  $\nu$  by means of Eq. (13), where the parameter  $t_n$  is now the difference between the values of  $\sigma_J$  in the two copies at RG step  $n$ .

Continuing this way we find the values shown in Table VIII for different HLs and we notice that they are compatible with each other within the statistical error. When compared to the estimate for the cubic Bravais lattice, though, only the WB one turns out to be nearly consistent with  $\nu = 2.45(15) \rightarrow 1/\nu = 0.408(25)$ .<sup>87</sup>

Proceeding as for the correlation length, and using the free energy  $f$  scaling law

$$f(t) = b^{-d} f(t') = b^{-d} f(b^x t) = b^{-nd} f(b^{nx} t)$$

critical index	MK Fig. 12	WB Fig. 4	Folded cube Fig. 11	Cubic Ref. 87
$\alpha$	-8.10(88)	-9.6(24)	-10.4(17)	-5.4(5)
$\beta$	3.37(30)	4.20(86)	4.27(60)	0.77(5)
$\gamma$	3.37(30)	1.35(28)	2.92(41)	5.8(4)
$\nu$	3.37(30)	3.25(66)	3.82(54)	2.45(15)
$\eta$	1	1.585...	1.236...	-0.375(10)

TABLE IX: Estimates for the physical exponents at transition SG-para for the Ising spin glass model.

with  $n$  such that  $b^{nx}t = t_0$  and  $t_0$  arbitrary but fixed, we have

$$f(t) = \left(\frac{t_J}{t_0}\right)^{\frac{d}{x}} f(t_0) \sim t^{\frac{d}{x}}. \quad (30)$$

Then from  $\partial^2 f / \partial t^2 \sim |t|^{-\alpha}$ , we obtain the scaling relation

$$\alpha = 2 - \frac{d}{x} = 2 - d\nu, \quad (31)$$

as well as

$$\begin{aligned} \beta &= \frac{\nu(1+\eta)}{2}, \\ \gamma &= (2-\eta)\nu, \end{aligned}$$

where

$$\eta = d + 2 - 2y_h = d - 2$$

since  $y_h = 2$  for all our HLs. The estimates of the physical exponents are reported in Tab. IX.

Concluding this section, on the FC lattice in Fig. 11 the estimates for the pure criticality are much closer to those on cubic lattice (although still not compatible), compared to MK in Fig. 12 and WB in Fig. 4. Also for the stiffness exponent of SG phase a remarkable improvement is observed: its estimate is compatible with its cubic value on FC lattice but on other HLs. For others quantities, though, at the disordered fixed point such improvement is unseen. In particular in the estimate for the  $\nu$  exponent at the SG-PM transition. We can argue that this is so because the stiffness exponent depends more on the local geometrical properties of the lattice, that is really improved moving to the FC lattice, whilst for the critical properties depends on longer distances, that are still dominated by the hierarchical backbone.

More stringent tests can be obtained for the Blume-Emery-Griffiths model, that we will analyze in the next section. For this system the inverse first order transition expected from mean-field theory<sup>32</sup> and simulation in finite dimension<sup>31</sup> is absent on the MK lattice in Fig. 12.<sup>35</sup>

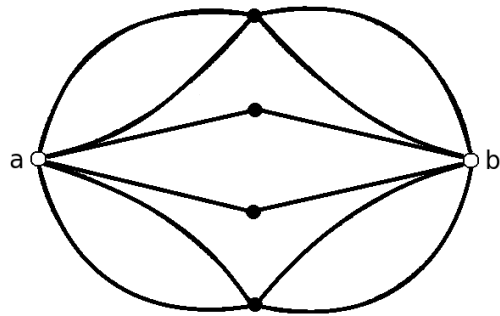


FIG. 16: MK lattice with fractal dimension  $d = 3.58$ , the same as WB lattice in Fig. 4.

#### IV. BLUME-EMERY-GRIFFITHS MODEL

We now move on to a different system, the Blume-Emery-Griffiths (BEG) model, originally devised to study the superfluidity transition and phase separation in  $\text{He}^3$ - $\text{He}^4$  mixtures.<sup>88</sup> The model is known to display, besides a second order phase transition, also a first order transition, both in the ordered case (between the PM and FM phases) and in the case with quenched disordered interactions (between the PM and SG phases).

The ordered model cases have been introduced and solved in the mean-field approximation in Refs. [88–90]. Finite dimensional analysis has been carried out by different means, e.g., series extrapolation techniques,<sup>91</sup> PSRG analysis,<sup>16</sup> Monte Carlo simulations,<sup>92</sup> effective-field theory<sup>93</sup> or two-particle cluster approximation.<sup>94</sup>

Extensions to quenched disordered models, both perturbing the ordered fixed point and in the regime of strong disorder, have been studied throughout the years by means of mean-field theory,<sup>36,95,96</sup> PSRG analysis on Migdal-Kadanoff hierarchical lattices,<sup>35,45</sup> and Monte Carlo numerical simulations.<sup>31,37,38,97</sup> The latter studies show that a critical transition line separates the SG and PM phases. Like in the mean-field cases, it consists of a second order transition terminating in a tricritical point from which a first order inverse transition starts.<sup>31</sup> Furthermore, a reentrance of the first order transition line is present for positive, finite values of the chemical potential of the holes,<sup>36</sup> yielding the so called inverse freezing phenomenon of an amorphous phase arresting itself in a blocked, solid-like state upon heating.

In absence of any purely ferromagnetic contribution, the PSRG approach on MK cells apparently does not show any first order phase transitions, nor any reentrance, as shown by Ozelik and Berker.<sup>35</sup> In this section we will deepen their analysis at  $p = 1/2$ , investigating the critical behavior on the MK lattices of Fig. 12 and Fig. 16, on the Wheatstone Bridge (WB) lattice of Fig. 4 and on the Folded Cube (FC) lattice of Fig. 11.

The BEG model with generic magnetic exchange in-

interaction is defined by the Hamiltonian (we use reduced variables)

$$-\beta\mathcal{H}(s) = \sum_{\langle ij \rangle} J_{ij} s_i s_j + K \sum_{\langle ij \rangle} s_i^2 s_j^2 - \Delta \sum_i s_i^2, \quad (32)$$

where  $s_i = \pm 1, 0$  and  $J_{ij}$  can be deterministic or quenched disordered and distributed according to some probability distribution. In the latter case, under PSRG transformation all renormalized interactions become quenched random and it is convenient to start the iteration using the more general form

$$\begin{aligned} -\beta\mathcal{H} &= \sum_{\langle ij \rangle} J_{ij} s_i s_j + \sum_{\langle ij \rangle} K_{ij} s_i^2 s_j^2 \\ &- \sum_{\langle ij \rangle} \Delta_{ij} (s_i^2 + s_j^2) - \sum_{\langle ij \rangle} \Delta_{ij}^\dagger (s_i^2 - s_j^2) \end{aligned} \quad (33)$$

The model is, further, defined, by the multivariable initial probability distribution of the interactions:

$$\begin{aligned} \mathcal{P}(J_{ij}, K_{ij}, \Delta_{ij}, \Delta_{ij}^\dagger) &= \frac{\delta(J_{ij} - J_0) + \delta(J_{ij} + J_0)}{2} \\ &\times \delta(K_{ij}) \delta(\Delta_{ij} - \Delta_0) \delta(\Delta_{ij}^\dagger) \end{aligned} \quad (34)$$

We notice that if  $\Delta_0 \ll -1$  the values  $s_i = 0$  are suppressed and the model tends to the Ising spin glass model analyzed in previous sections:

$$-\beta\mathcal{H}(\{J_{ij}\}; \{s_i\}) = \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

with  $s_i = \pm 1$  and

$$P(J_{ij}) = \int dK_{ij} d\Delta_{ij} d\Delta_{ij}^\dagger \mathcal{P}(J_{ij}, K_{ij}, \Delta_{ij}, \Delta_{ij}^\dagger).$$

Decimating the inner sites at a given hierarchical cell with fixed outer sites  $s_a$  and  $s_b$ , and using the up-down symmetry of the Hamiltonian, the relations for the renormalized interactions imposed by the conservation of the partition function, cf. Eq. (4), can be written similarly to Eqs. (5)-(4) as

$$\begin{aligned} J_R &= \frac{1}{2} \log \left( \frac{x_{++}}{x_{+-}} \right), \\ K_R &= \frac{1}{2} \log \left( \frac{x_{++} x_{+-} x_{00}^2}{x_{0+}^2 x_{+0}^2} \right), \\ \Delta_R &= \frac{1}{2} \log \left( \frac{x_{00}^2}{x_{+0} x_{0+}} \right), \\ \Delta_R^\dagger &= \frac{1}{2} \log \left( \frac{x_{+0}}{x_{0+}} \right), \end{aligned} \quad (35)$$

where  $x_{s_a s_b}$  are the Boltzmann factors, cf. Eq. (4). In

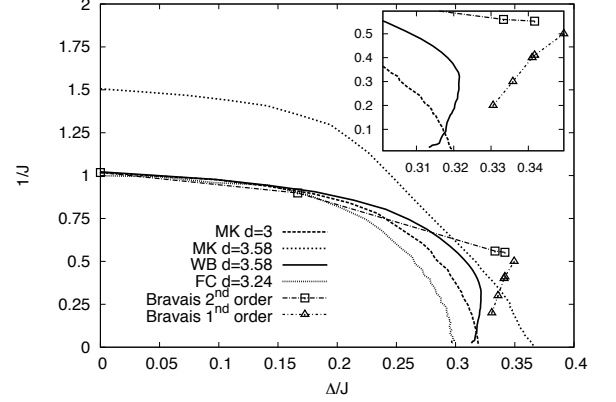


FIG. 17: Phase diagram for the BEG model on the MK( $d = 3$ ) lattice in Fig. 12, WB lattice in Fig. 4, FC lattice in Fig. 11 and simulations<sup>31</sup>. The line for the MKs and WB are obtained with  $M = 10^6$ , whilst for the FC we use  $M = 5 \cdot 10^4$ . Inset: detail of the reentrance region for WB and 3D cubic lattices, compared to the MK HL line (no reentrance).

the  $T = 0$  limit these relations become

$$\begin{aligned} 2J_R &= \max[-\beta\mathcal{H}(1, 1, s)] - \max[-\beta\mathcal{H}(1, -1, s)] \\ 2K_R &= \max[-\beta\mathcal{H}(1, 1, s)] + \max[-\beta\mathcal{H}(1, -1, s)] \\ &- 2\{ \max[-\beta\mathcal{H}(1, 0, s)] - \max[-\beta\mathcal{H}(0, 1, s)] \} \\ &+ 2\max[-\beta\mathcal{H}(0, 0, s)] \\ 2\Delta_R &= 2\max[-\beta\mathcal{H}(0, 0, s)] \\ &- \max[-\beta\mathcal{H}(1, 0, s)] - \max[-\beta\mathcal{H}(0, 1, s)] \\ 2\Delta_R^\dagger &= \max[-\beta\mathcal{H}(1, 0, s)] - \max[-\beta\mathcal{H}(0, 1, s)] \end{aligned} \quad (36)$$

The phase diagrams relative to the different hierarchical lattices are shown in Fig. 17. They are obtained representing the probability distributions by a pool of  $M = 10^5$  interaction quadruples  $(J, K, \Delta, \Delta^\dagger)$  and every evolution is repeated  $N_S = 10$  times. As worked out in Sec. III paramagnetic, ferromagnetic and spin glass phases are determined from the analysis of the PSRG flux of  $J = \langle J_{ij} \rangle$  and  $\sigma_J^2 = \langle J_{ij}^2 \rangle - \langle J_{ij} \rangle^2$ .

In all cases we obtain a second order transition between paramagnetic and spin glass phases, with all points on the transition line attracted by a unique fixed distribution at  $\Delta \rightarrow -\infty$  (see Fig. 18), thus belonging to the same universality class of the SG-PM transition in the Ising spin glass investigated in the previous section.

Also studying hierarchical lattices much more complex than MK, the first order transition typical of the BEG model on Bravais lattice is not found, so we have a strong indication that this is an intrinsic limit of hierarchical lattices, and not only of the MK kind of cells.

It is important to emphasize, however, that on the WB  $d \approx 3.58$  lattice we obtain a feature not present for FC

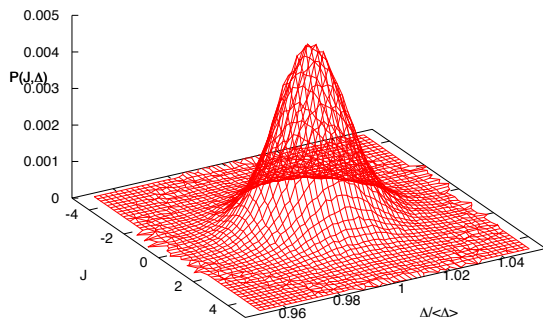


FIG. 18: (Projection of unstable fixed distribution  $\mathcal{P}(J, K, \Delta, \Delta^\dagger)$  for BEG model at the transition SG-para on the WB lattice in Fig. 4. In the distribution  $\Delta$  is runaway to minus infinity exponentially, while the other interactions remain finite and the distribution retains this shape. The shape of relative distribution on the MK and FC lattices is very similar.

lattice and for MK lattices, neither in fractal dimension 3 (cf. Fig. 12) nor in  $d \approx 3.58$  (cf. Fig. 16): an inverse transition between spin glass and paramagnet. Evidence for the reentrance is clearly obtained using the  $T = 0$  Eqs. (36). Inverse freezing is predicted in mean-field theory<sup>36</sup> and found in 3D numerical simulations.<sup>31</sup>

## V. CONCLUSIONS

We have provided a review of the standard methods to develop PSRG on hierarchical lattices. We stress how on one side this analysis can be useful to test general results (e.g. Nishimori conjecture, Harris criterion), and on the other side it is far easier and faster than Montecarlo simulations on Bravais lattice.

We have used these methods to investigate the Random Field Ising Model (RFIM), the Random Bond Ising Model (RBIM) and the Blume-Emery-Griffiths (BEG) model defined on several MK and non-MK hierarchical lattices, obtaining the whole phase diagrams and critical exponents.

The RFIM has been analyzed on a non-MK (WB in Fig. 4) for the first time, and we have shown that the bimodal and Gaussian disordered cases belong to the same universality class.

The RBIM in  $d < 2.5$  has been analyzed on the folded square HLs family for  $b = 3$  (Fig. 5) and  $b = 5$  (Fig. 6), where we find that the Nishimori conjecture fails and the disorder is irrelevant.

The RBIM in  $d > 2.5$  has been analyzed on the folded cube (Fig. 11) and compared to MK (Fig. 12) and WB (Fig. 4), where we obtain the critical exponents for the

SG-PM transition and also show that the Harris criterion holds.

The BEG model has been analyzed on non-MK HLs for the first time, WB (Fig. 4) and folded cube (Fig. 11), and compared to MK with  $d = 3$  (Fig. 12) and  $d \approx 3.58$  (Fig. 16), and in all the cases the first order transition of the Bravais lattice<sup>31,36</sup> is absent.

Our results provide a clue to the possibility of obtaining approximations of models on regular lattice by a the similar models on hierarchical lattice.

We show that it is possible to obtain a good picture of the actual phase diagram, but more difficult a proper determination of critical exponents.

By introducing more complex elementary cells, with some non trivial internal structure, one hopes of capturing the local geometrical properties of the bonds. At least part of it. For pure models, although it is not a systematic approximation, a general improvement is obtained using more complex unit cells, that locally mimic better the connectivity of the Bravais lattice. In the disordered case, instead, the situation is less definite, and no net improvement is observed: the WB (Fig. 4) proves to be the slightly most reliable (generally quantitatively better than the more complex folded cube in Fig. 11), and in particular shows the expected inverse transition for the BEG model, but we can not give a general explanation for this.

In particular, the fractal dimension seems to play a minor role, as the three dimensional regular lattice is better approximated by the WB with  $d \sim 3.58$ , compared to  $d = 3$  MK (Fig. 12) and  $d \sim 3.24$  folded cube (Fig. 11), while the  $d \sim 3.58$  MK in Fig. 16 is the worst approximation by far.

On the other hand, the scaling factor  $b$  has the known role to determine if the antiferromagnetic order can be preserved, as it is possible only when  $b$  is odd so that negative interactions in the unit cell involve negative interactions between the external sites (and this leads to a phase diagram symmetric under the inversion of the bonds sign). Our results indicates that this feature does not play a crucial role in the disordered systems (at least until the negative bonds become dominant). The two investigated HLs with an even  $b$ , the WB and the  $d \sim 3.58$  MK, indeed, appear to be, respectively, the best and the worst in approximating the phase diagram of the models on regular lattice. The only case in which the folded cube lattice provides a remarkable improvement with respect to the WB lattice is in the estimate of the SG stiffness exponent. A more structured inner connectivity could, thus, become important at low temperatures.

In conclusion, our results show that the approximation on HL is particularly poor for disordered systems, and strongly suggest that its limitations are intrinsic to the hierarchical nature. The most striking case is the lack of first order transition in BEG. Indeed, using more complex HLs we may have a better treatment of short distances, i.e., short loops, but longer distances seem to be definitely dominated by the hierarchical backbone.



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